Vertex-disjoint directed and undirected cycles in general digraphs

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Abstract

The *dicycle transversal number* $\tau(D)$ of a digraph D is the minimum size of a *dicycle transversal* of D, i. e. a set $T \subseteq V(D)$ such that $D - T$ is acyclic. We study the following problem: Given a digraph D, decide if there is a dicycle B in D and a cycle C in its underlying undirected graph $UG(D)$ such that $V(B) \cap V(C) = \emptyset$. It is known that there is a polynomial time algorithm for this problem when restricted to strongly connected graphs, which actually finds B, C if they exist. We generalize this to any class of digraphs D with either $\tau(D) \neq 1$ or $\tau(D) = 1$ and a bounded number of dicycle transversals, and show that the problem is $\mathcal{NP}$-complete for a special class of digraphs D with $\tau(D) = 1$ and, hence, in general.

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1 Introduction

All graphs and digraphs are supposed to be finite, and they may contain loops or multiple arcs or edges. Notation follows [1], and we recall the most relevant concepts here. In order to distinguish between directed cycles in a digraph D and cycles in its *underlying graph* $UG(D)$ we use the name *dicycle* for a directed cycle in D and *cycle* for a cycle in $UG(D)$. Whenever we consider a (directed) path P containing vertices a, b such that a precedes b on P, we denote by $P[a, b]$ the subpath of P which starts in a and ends in b. Similarly, we denote by $P(a, b]$, $P[a, b)$, and $P(a, b)$, respectively, the subpath that starts in the successor of a on P and ends in b, starts in a and ends in the predecessor of b, and starts in the successor of a on P and ends in the predecessor of b, respectively. The same notation applies to dicycles.

An *in-tree* (*out-tree*) rooted at a vertex r in a digraph D is a tree in $UG(D)$ whose arcs are oriented towards (away from) the root in D.

A digraph D is *acyclic* if it does not contain a dicycle, and it is *intercyclic* if it does not contain two disjoint dicycles. A *dicycle transversal* of D is a set S of vertices of D such that $D - S$ is acyclic, and the *dicycle transversal number* $\tau(D)$ is defined to be the size of a smallest dicycle transversal. McCuaig characterized the intercyclic digraphs of minimal in- and out-degree at least 2 in terms of their dicycle transversal number and designed a polynomial time algorithm that, for any digraph, either finds two disjoint cycles or a structural certificate for being intercyclic [7].

Theorem 1 *[7] There exists a polynomial time algorithm which decides whether a given digraph is intercyclic and finds two disjoint cycles if it is not.*

The undirected graphs without two disjoint cycles have been characterized by Lovász [6], generalizing earlier statements of Dirac for the 3-connected case [4]. The characterization again implies a polynomial algorithm for finding such cycles if they exists.

Here we are concerned with the following problem.

Problem 1 *Given a digraph D, decide if there is a dicycle B in D and a cycle C in $UG(D)$ with $V(B) \cap V(C) = \emptyset$.*

The motivation for studying this problem comes from [2] where a mixed variant of the subdigraph homeomorphism problem has been studied. The problem of deciding if, for a given digraph D and $b, c \in V(D)$, there exist disjoint dicycles B, C in D with $b \in V(B)$ and $c \in V(C)$ is known to be $\mathcal{NP}$-complete by the classic dichotomy of Fortune, Hopcroft, and Wyllie on the *fixed directed subgraph homeomorphism problem* [5]: For some pattern digraph H, not part of the input, we want to decide for an input digraph D and an injection f from $V(H)$ to $V(D)$ if we can extend f on $V(H) \cup A(H)$ such that every loop at x maps to a cycle of D containing $f(x)$, every arc xy with $x \neq y$ maps to an $(f(x), f(y))$-path, and the resulting paths and cycles are *internally disjoint*, i.e. no internal vertex of either object is a vertex of another one[1]. The dichotomy then states that the problem is solvable in polynomial time if the arcs of H have the same initial vertex or if they have the same terminal vertex, and is $\mathcal{NP}$-complete in all other cases [5].

In [2], an extension of this has been studied, where H might be a *mixed graph*, having both arcs and edges, and the edges of H are asked to be mapped to cycles and paths of $UG(D)$ [2].[2] We found it surprising that, as a consequence of

[1] Where, in the case of a cycle C of D assigned to a loop of H at x, we consider its internal vertices to be all but $f(x)$.

[2] We are always assuming that D and $UG(D)$ have the same set of vertices and arcs, respectively, i.e. they differ only by means of incidence relations.

the resulting dichotomy, the problem is already $\mathcal{NP}$-complete as soon as there is both an arc and an edge in the pattern graph. In particular, the problem of deciding whether for a digraph D and $b, c \in V(D)$ there exists a cycle B in D and a cycle C in $UG(D)$ with $b \in V(B)$, $c \in V(C)$, and $V(B) \cap V(C) = \emptyset$, is $\mathcal{NP}$-complete. The proof shows that even the weaker problem to decide whether for a digraph D and $c \in V(D)$ there exists a cycle B in D and a cycle $C \in UG(D)$ with $c \in V(C)$ and $V(B) \cap V(C) = \emptyset$ is $\mathcal{NP}$-complete, even if we are assuming that, in addition, D is strongly connected.

So the question arised what happens if we do not prescribe vertices at all, leading to Problem 1.

The first two authors showed in [3] that Problem 1 is solvable in polynomial time when D is strongly connected. The solution turned out to be more complex than expected, and builds on McCuaig's results on intercyclic digraphs [7], Thomassen's results on 2-linkages in acyclic digraphs [9], and a new reduction algorithm for digraphs with dicycle transversal number one.

Theorem 2 *[3] There is a polynomial algorithm for Problem 1 restricted to strongly connected digraphs. Furthermore, one can find the desired cycles in polynomial time if they exist.*

In this paper, based on the complete characterization from [3] of those strongly connected digraphs with dicycle transversal number 2 which are no-instances for Problem 1 (Theorem 4), we will show that there is a polynomial time algorithm for Problem 1 restricted to digraphs with dicycle transversal number at least 2. After this we show that Problem 1 is $\mathcal{NP}$-complete for digraphs with $\tau(D) = 1$, and, hence, $\mathcal{NP}$-complete in general.

The case $\tau(D) \geq 3$ is easily dealt with due to the following result from [3].

Theorem 3 *[3] If D is a strongly connected digraph with $\tau(D) \geq 3$ then there is a dicycle B in D and a cycle C in $UG(D)$ with $V(B) \cap V(C) = \emptyset$, and we can find such cycles in polynomial time.*

Since a digraph with at least two non-trivial strong components of size greater than one has two disjoint dicycles, we get, as an immediate consequence:

Corollary 1 *There exists a polynomial time algorithm for Problem 1 restricted to digraphs with dicycle transversal number at least 3, which finds the desired cycles.*

Trivially, acyclic digraphs are no-instances to Problem 1, so let us assume that the digraphs D under consideration have at least one dicycle. McCuaig's algorithm from [7] finds two disjoint dicycles in D if they exist. If they do not

exist we know that the digraphs D under consideration have exactly one non-trivial strong component D', where $\tau(D') = \tau(D) = \{1, 2\}$. We then apply the algorithm from [3] to D'; if D' is a yes-instance to Problem 1 then so is D, so that we can assume that D' is a no-instance to Problem 1. For $\tau(D) = 2$, we employ the complete characterization of no-instances in [3] and derive a polynomial time algorithm which takes the (undirected) cycles in D but not in D' into account to produce a correct answer. If $\tau(D) = 1$ then we give an algorithm with running time $\ell(D)^{k(D)} \cdot p(|V(D)|)$, where p is a polynomial, $k(D)$ is the number of dicycle transversal vertices of D, and $\ell(D)$ is the maximum number of disjoint paths between a pair of distinct transversal vertices. Since $\ell(D) \leq |V(D)|$, this is a polynomial time algorithm if we are in any class of digraphs with $\tau(D) = 1$ and a constantly bounded number of dicycle transversals. In Section 4 we give a proof that Problem 1 is $\mathcal{NP}$-complete for a certain class of digraphs with dicycle transversal number 1 (and hence in general) by providing a two-step reduction from 3SAT to Problem 1.

2 Strongly connected digraphs with dicyle transversal number 2

In this section we describe the characterization in [3] of the strongly connected no-instances to Problem 1. They fall into three infinite classes called vaults, multiwheels, and trivaults.

We start by describing the vaults. Let $\ell \geq 5$ be odd, let $P_0, \ldots, P_{\ell-1}$ be disjoint nonempty paths, and, for each $i \in \{0, \ldots, \ell - 1\}$, let a_i be the initial vertex, d_i be the terminal vertex, and b_i, c_i be vertices of P_i such that either $b_i c_i$ is an arc on P_i or $b_i = c_i \in \{a_i, d_i\}$. Suppose that D is obtained from the disjoint union of the P_i by

(i) adding at least one arc from some vertex in $P_i[c_i, d_i]$ to some vertex from $P_{i+1}[a_{i+1}, b_{i+1}]$ (multiarcs may occur), and

(ii) adding a single arc from d_i to a_{i+2}, for all $i \in \{0, \ldots, \ell - 1\}$,

where the indices are taken modulo ℓ. Any digraph of such a form is called a *vault*, and the P_i are called its *walls*. We say that the vault D has a *niche*, if there exist arcs pq, rs from some P_i to P_{i+1} such that p occurs before r on P_i and q occurs after s on P_{i+1}. In that case,

$$P_i[a_i, p]P_{i+1}[q, d_{i+1}]P_{i+3}[a_{i+3}, d_{i+3}] \ldots P_{i-2}[a_{i-2}, d_{i-2}]a_i$$

is a dicycle of D, disjoint from the cycle of $UG(D)$ constituted by the path

$$P_i[r, d_i]P_{i+2}[a_{i+2}, d_{i+2}]P_{i+4}[a_{i+4}, d_{i+4}] \ldots P_{i-1}[a_{i-1}, d_{i-1}]P_{i+1}[a_{i+1}, s]$$

and the arc rs. Figure 1 shows a vault with $\ell = 5$, where all paths $P_i[a_i, b_i]$ or $P_i[c_i, d_i]$ have seven vertices; the grey areas indicate the set of arcs connecting

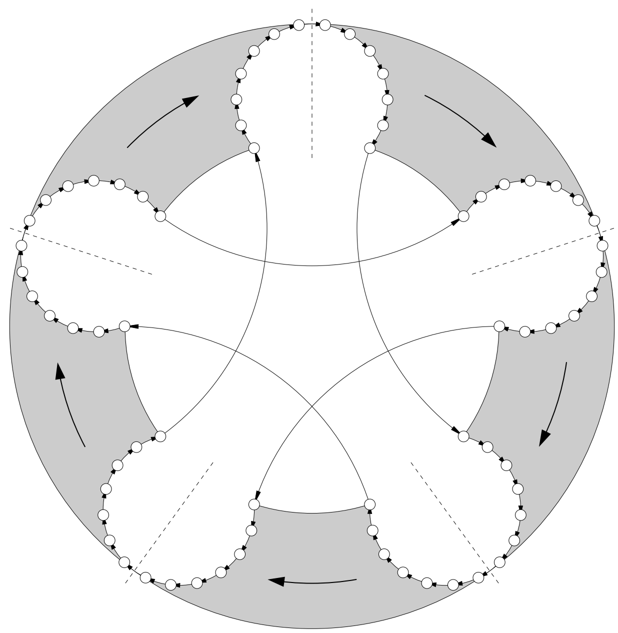

Figure 1: A typical vault. The five central arcs must have multiplicity 1 and are the only arcs from P_i to P_{i+2}.

$P_i[c_i, d_i]$ to $P_{i+1}[a_{i+1}, b_{i+1}]$, a niche would correspond to a pair of arcs which can be drawn without crossing in such an area. Vaults are strongly connected digraphs as they have a spanning dicycle. They may contain vertices of both in- and out-degree 1, but, as they occur only as internal vertices of the P_i, we deduce that every vault D is a subdivision of a vault $\tilde{D}$ without vertices of in- and out-degree 1, where $\tilde{D}$ has a niche if and only if D has.

A *multiwheel* MW_p is obtained from a directed cycle $c_0c_1 \dots c_{p-1}c_0$, $p \geq 3$, by adding a new vertex v and adding, for each $i \in \{0, \dots, p-1\}$, ℓ_i arcs from v to c_i and k_i arcs from c_i to v where $\ell_i + k_i \geq 1$. A *split multiwheel* SMW_p is obtained from a multiwheel MW_p by replacing the central vertex v by two vertices v^+, v^-, adding the arc v^-v^+, and letting all arcs entering (leaving) v in MW_p enter (leave) v^- (v^+). See Figure 2. The vertices v or v^+, v^- are called the *central vertices* of the multiwheel or split multiwheel, respectively.

A *trivault* is obtained from six disjoint digraphs R_i, L_i, $i \in \{0, 1, 2\}$, where each R_i is either a nontrivial out-star with root b_i or a (b_i, x_i)-path and each L_i is either a nontrivial in-star with root c_i or a (y_i, c_i)-path, as follows:

(i) for each $i \in \{0, 1, 2\}$ either add a single arc from c_i to b_i or identify b_i, c_i,

(ii) for distinct $i, j \in \{0, 1, 2\}$, if R_i is a nontrivial out-star and L_j is a nontrivial in-star, add a single arc from each leaf of R_i to c_j and from b_i to

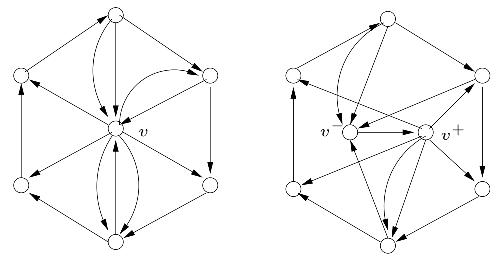

Figure 2: The left part shows a multiwheel with center v and the right one the split multiwheel obtained from that by splitting v into v^-, v^+.

every leaf of L_j and an arbitrary number of arcs (possibly 0) from b_i to c_j,

(iii) for distinct $i, j \in \{0, 1, 2\}$, if R_i is a nontrivial out-star and L_j is a path, select $v \in L_j$ and add a single arc from each leaf of R_i to v, at least one arc from b_i to y_j, and an arbitrary number of arcs (possibly 0) from b_i to each $z \in L_j[y_j, v]$,

(iv) similarly, for distinct $i, j \in \{0, 1, 2\}$, if R_i is a path and L_j is a nontrivial in-star, select $v \in R_i$ and add a single arc from v to each leaf of L_j, at least one arc from x_i to c_j, and an arbitrary number of arcs (possibly 0) from each $z \in R_i[v, x_i]$ to c_j, and

(v) if, for distinct $i, j \in \{0, 1, 2\}$, R_i, L_j are paths, then add at least one arc from x_i to some vertex of L_j, and at least one arc from some vertex of R_i to y_j, and add an arbitrary number of arcs (possibly 0) from each $z \in R_i$ to each $w \in L_j$.

Figure 3 shows a typical trivault. Allowing $\ell = 3$ in the definition of vaults will produce other trivaults, but not all. We say that a trivault has a *niche* if there are distinct $i, j, k \in \{0, 1, 2\}$ such that either

(a) R_i, L_j are paths and there are arcs pq, rs such that p occurs before r on R_i and q occurs after s on L_j, or

(b) R_i is a path, containing an in-neighbor x of L_k such that there are at least two arcs from $R_i(x, x_i]$ to L_j, or

(c) L_i is a path containing an out-neighbor y of R_k such that there are at least two arcs from R_j to $L_i[y_i, y)$.

Observe that every trivault is strongly connected. It might contain a vertex of in- and out-degree 1; however, this is either in some path $R_i - x_i$ or in some path $L_i - y_i$, and contracting any arc (on that path) incident with it produces, consequently, a trivault again; this smaller trivault will have a niche only if the original one had a niche. Hence we can consider every trivault as a subdivision

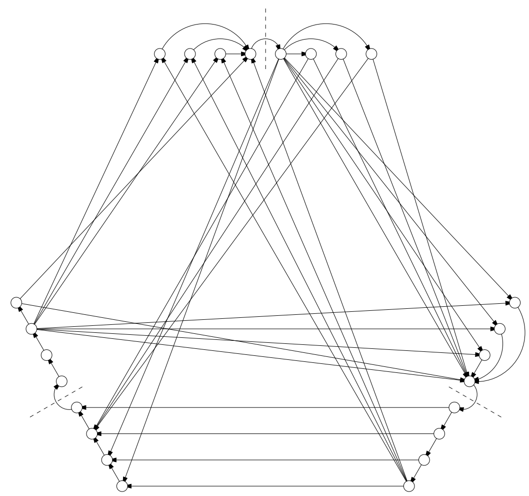

Figure 3: A typical trivault. The dotted lines separate the two parts of each display member.

of a trivault without vertices of in- and out-degree 1, which has a niche if and only if the primal trivault had.

Now we are ready to state the characterization from [3] of the strongly connected no-instances.

Theorem 4 *[3] Let $D = (V, A)$ be a strongly connected digraph with dicycle transversal number 2. In polynomial time we can either find a cycle B in D and a cycle C in $UG(D)$ with $V(B) \cap V(C) = \emptyset$ or show that D has no such cycles in which case D satisfies one of the following.*

(i) D is a subdivision of a vault without a niche.

(ii) D is a subdivision of either a multiwheel or a split multiwheel.

(iii) D is a subdivision of a trivault without a niche.

Furthermore, if D satisfies one of (i)-(iii), we can produce a certificate for this in polynomial time.

In order to obtain a certificate that a given strongly connected digraph D with $\tau(D) = 2$ is in fact a no-instance, we first reduce to an equivalent instance $\bar{D}$ which has minimum in and out-degree 2 and then apply the following theorem from [3].

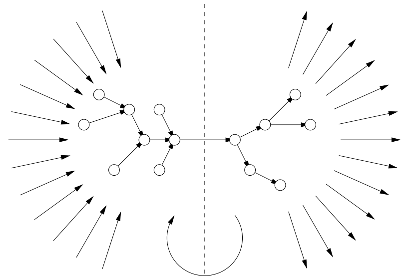

Figure 4: A typical display member P_v. Arcs not in $A(P_v)$ but incident with some vertex from $V(P_v)$ will start in its out-tree or terminate in its in-tree (or both). The in- and out-trees are displayed on the left and right hand side of the drawing, respectively. Instead of being adjacent as indicated, their respective roots might be the same ("thought of being on the dashed line").

Theorem 5 *[3] Let D_0 be an intercyclic digraph with $\tau(D_0) = 2$ and minimal in- and out-degree at least 2. Then there is a dicycle B in D_0 and a cycle C in $UG(D_0)$ with $V(B) \cap V(C) = \emptyset$ if and only if D_0 is not among the following digraphs.*

(i) A complete digraph on 3 vertices (with arbitrary multiplicities).

(ii) A digraph obtained from a cycle Z on at least 3 vertices by adding a new vertex a and at least one arc from a to every $b \in V(Z)$ and at least one arc from every $b \in V(Z)$ to a.

(iii) A digraph obtained from a cycle Z of odd length ≥ 5 by taking its square and adding an arbitrary collection of arcs parallel to those of Z.

A *reduction* D' of a digraph D is obtained from D by contracting arcs e which are the unique out-arc at its initial vertex or the unique in-arc at its terminal vertex as long as it is possible. It is clear that every vertex v of the reduction D' either corresponds to a nonempty set of arcs which form a subdigraph P_v of D where P_v is connected in $UG(D)$, or is a vertex of D, forming the arcless digraph P_v; we call the family $(P_v)_{v \in V(D')}$ the *display* of the reduction.[3]

Lemma 1 *[3] Let D be a strongly connected digraph without vertices of both in- and out-degree 1. Then*

(i) there is only one reduction D^R, up to the labelling of the newly introduced vertices in the contraction process,

[3] We took the symbol P_v for the display members, as they turn out to be paths in many cases.

(ii) for its display $(P_v)_{v\in V(D^R)}$, each P_v is either the union of an in-tree L_0 and an out-tree R_0 which have only their root in common, or the union of an in-tree L_0 and an out-tree R_0 disjoint from L_0 plus an additional arc from the root of L_0 to the root of R_0, such that, in both cases, every arc in $A(D)-A(P_v)$ starting in P_v starts in R_0 and every arc in $A(D)-A(P_v)$ terminating in P_v terminates in L_0. (See Figure 4.)

3 Digraphs D with $\tau(D)=2$

We now look at the case that our input digraph D to Problem 1 has dicycle transversal number 2. As we have mentioned in the introduction, we may assume that D is not strongly connected and has exactly one non-trivial component D', where, moreover, D' is a no-instance. By Theorem 4, D' is either the subdivision of a niche-free vault, a multi-wheel (splitted or not), or a niche-free trivault.

Clearly, if $UG(D-D')$ contains a cycle then D is a yes-instance. Hence we may assume that $UG(D-D')$ is a forest, that is, all connected components of $UG(D-D')$ are trees. We observe that if D is a yes-instance then there exists a cycle C in $UG(D)$ disjoint from some dicycle B in D' such that C traverses every component H of $UG(D-D')$ at most once (for if C traverses H then we consider a component P of $C-V(H)$ and the — not necessarily distinct — neighbors h,h' of the endvertices of P in H on C, and replace the h,h'-path $C-V(P)$ with the h,h'-path in H as to obtain a cycle C' disjoint from B traversing H only once). Thus we loose no information by contracting every connected component of $D-D'$ to a single vertex, and reorienting all arcs between a vertex of $D-D'$ and D' so that they all terminate in D'. Hence $D-D'$ consists of independent vertices, which we call the *external vertices*. Since $\tau(D')=2$ the following holds:

Lemma 2 *If there are parallel arcs from $D-D'$ to D' then D is a yes-instance.*

We further simplify the problem by observing that each of the following operations can be applied to D without changing a no-instance into a yes-instance or vice versa. We repeat doing any one of these as long as possible, while always calling the resulting graph D and its non-trivial strong component D' and observing that the dicycle transversal number does not change either.

(i) If there is more than one arc from u to v check if $\{u,v\}$ is a dicycle transversal. If not, then D is a yes-instance (take uvu as the undirected cycle). Otherwise we delete all but one copy of uv.

(ii) Delete all external vertices with degree at most one (they are on no cycle).

(iii) Contract the outgoing arc of a vertex v with $d^-_D(v)=1=d^+_D(v)$.

We now analyze connections between pairs of vertices in D' and external vertices. Our actual setup guarantees that any undirected cycle C (partly) certifying a yes-instance must use at least one external vertex. It is possible to show that if D is a yes-instance, then we can choose C such that it contains at most two external vertices. However, we will illustrate this only for the vault-case, whereas for multiwheels and trivaults it is much easier to control all possible dicycles (as a matter of the method, the resulting algorithms are more of a brute force type).

Vaults.

A pair $(\{u, v\}, \alpha)$ is called a *k-clasp* if α is an external vertex, u, v are neighbors of α, and there exists a cycle C^* in $UG(D)$ containing u, v, α and at most k external vertices such that there exists a dicycle B^* in $D - V(C)$. By definition, there cannot be a 0-clasp, and by what we have seen before, u, v need to be distinct. Observe that there exists a k-clasp if and only if D is a yes-instance.

By Theorem 4, D' is a subdivision of some graph D'_0, where D'_0 is a vault without a niche, a multiwheel or a split multiwheel, or a trivault without a niche. We proceed by distinguishing cases accordingly. Given an arc $pq \in D'_0$, we denote by $\widehat{pq}$ the corresponding subdivision dipath in D' and call it, for brevity, a *link*. A link of length 1 is called trivial.

Let us first treat the case that D' is a subdivision of a niche-free vault D'_0, with walls P_i, and let a_i, b_i, c_i, d_i be vertices on P_i as in the definition of a vault, $i \in \{0, ..., \ell - 1\}$ (all indices modulo ℓ). We may assume that consecutive vertices on P_i are not subdivided (so that the P_i are paths in D', too). If $\widehat{d_i a_{i+2}}$ is non-trivial then we enlarge the wall P_i by $\widehat{d_i a_{i+2}} - a_{i+2}$ and redefine d_i accordingly. Hence we may assume that non-trivial links always connect consecutive walls.

Lemma 3 *There is always a directed cycle avoiding any prescribed wall, but there is no directed cycle avoiding two consecutive walls.*

Proof. It is easy to check that the subdigraph consisting of the walls P_{i-1}, P_{i+1}, P_{i+3}, $\ldots$, P_{i-4}, P_{i-2} and all links between them contains a directed cycle avoiding P_i. On the other hand a directed cycle avoiding walls P_i, P_{i+1}, if it existed, could not contain vertices of P_{i+2}, because $V(P_{i+2})$ has no in-degree in $D' - V(P_i) \cup V(P_{i+1})$. Repeating this argument inductively one sees that no wall could be part of the cycle, hence such a cycle cannot exist. $\square$

Lemma 4 *If u, v are distinct neighbors of an external vertex α and u, v are either on the same wall or on distinct non-consecutive walls then $(\{u, v\}, \alpha)$ is a 1-clasp.*

Proof. If u and v are on the same wall P_i, then by Lemma 3, there is a directed cycle avoiding P_i, which is therefore disjoint from the undirected cycle

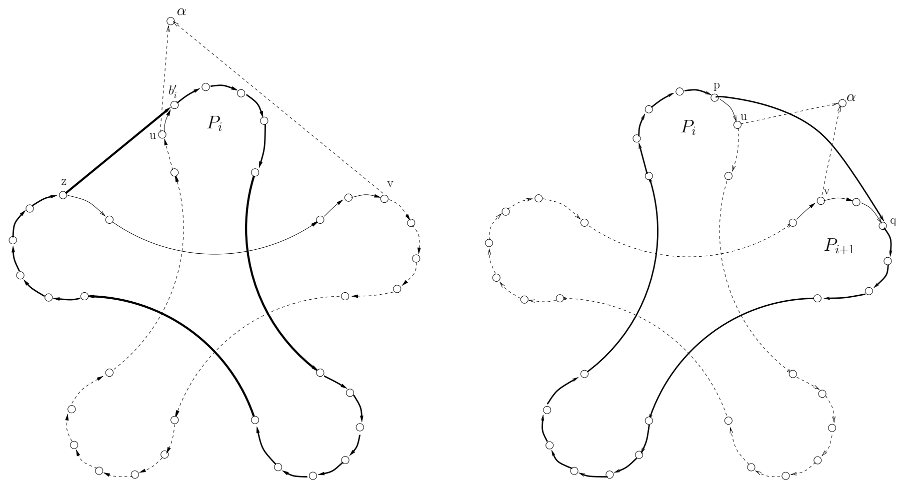

Figure 5: Possible 1-clasps formed by an external vertex with two neighbours on a vault. The directed cycle is indicated in bold and the other cycle by dashed arcs.

containing α, u, v and using only vertices from $V(P_i) \cup \{\alpha\}$. If u and v are not on the same P_i it is possible to relabel everything in such a way that u is on the wall P_0 and v is on wall P_{2k}, where $2k > 0$ and $2k < \ell - 1$. The undirected cycle $P_0[u, d_0]P_2P_4 \dots P_{2k-2}P_{2k}[a_{2k}, v]\alpha u$ is therefore disjoint from any directed cycle contained in the subdigraph induced by $P_1, P_3, P_5, \dots, P_{\ell-4}, P_{\ell-2}, P_{\ell-1}$ and all the links between them. $\square$

Let b_i' (c_i') be the last (first) vertex on P_i such that there exists a link from a vertex in P_{i-1} to b_i' (from c_i' to a vertex in P_{i+1}). A pair $(\{u, v\}, \alpha)$ is a *pin* if α is an external vertex, u, v are neighbors of α, there exists an $i \in \{0, \dots, \ell - 1\}$ such that u is in $P_i[b_i', d_i]$ and v is in $P_{i+1}[a_{i+1}, c_{i+1}']$ and such that there is no link $\widehat{pq}$ with p in $P_i[a_i, u)$ and $q \in P_{i+1}(v, d_{i+1}]$.

The following Theorem classifies all sets $\{u, v\}$ of two distinct vertices from D' with a common external neighbor α: Either $\{u, v\}$ is a dicycle transversal of D', or $(\{u, v\}, \alpha)$ is a 1-clasp. (Hence if there is no 1-clasp in D at all then all such $\{u, v\}$ are dicycle transversals of D', so that we cannot find a k-clasp for any k, and hence D is a no-instance for Problem 1.)

Theorem 6 *Let $u \neq v$ be vertices from D' with a common external neighbor α.*

(i) $(\{u, v\}, \alpha)$ is a pin if and only if $\{u, v\}$ is a dicycle transversal of D'.

(ii) $(\{u, v\}, \alpha)$ is not a pin if and only if $(\{u, v\}, \alpha)$ is a 1-clasp.

Proof. Since it is not possible that $\{u, v\}$ is a dicycle transversal of D' while $(\{u, v\}, \alpha)$ is a 1-clasp, it suffices to prove the only-if-parts of (i) and (ii).

For (i), suppose that $(\{u,v\},\alpha)$ is a pin, and let i be as in the definition of a pin. We show that the walls P_i and P_{i+1} containing u and v, respectively, cannot be part of a dicycle which avoids u,v and then use Lemma 3 to conclude that $\{u,v\}$ is a dicycle transversal. If we remove u then, as u does not occur before b'_i, the path starting from u's out-neighbour on P_i (if one exist) and ending at d_i has in-degree zero and hence cannot be contained in a directed cycle, so we can remove it during our the search. Symmetrically, the path starting from a_{i+1} and ending at v's in-neighbour (if one exist) on P_{i+1} has out-degree zero and can be removed. At this stage the set consisting of the remaining vertices on P_i and the set consisting of the remaining vertices on P_{i+1} have zero out-degree and in-degree respectively, hence they cannot be part of a dicycle. Now Lemma 3 implies that $\{u,v\}$ is a dicycle transversal.

For (ii), suppose that $(\{u,v\},\alpha)$ is not a pin. We prove that $(\{u,v\},\alpha)$ is a 1-clasp. First consider the case of u,v both being on walls: If u,v are on the same wall or on distinct non-consecutive walls, then Lemma 4 guarantees that $(\{u,v\},\alpha)$ is a 1-clasp. So let us assume that there exists an $i \in \{0,\ldots,\ell-1\}$ such that u is on P_i and $v \in P_{i+1}$. If u comes before b'_i on P_i then there is a link $\widehat{zb'_i}$, with $z \in P_{i-1}$, and the directed cycle $\widehat{zb'_i}P_i(b'_i,d_i]P_{i+2}...P_{i-1}[a_{i-1},z]$ is disjoint from the undirected cycle $P_{i+1}[v,d_{i+1}]P_{i+3}....P_i[a_i,u]\alpha v$ (see left part of Figure 5). Symmetrically if v comes after c'_{i+1} then there is a link $\widehat{c'_{i+1}z'}$, with $z' \in P_{i+2}$ and the directed cycle $\widehat{c'_{i+1}z'}P_{i+2}(z',d_{i+2}]P_{i+4}...P_{i+1}[a_{i+1},c'_{i+1}]$ is disjoint from the undirected cycle $P_{i+1}[v,d_{i+1}]P_{i+3}...P_i[a_i,u]\alpha v$. Hence we may assume that u is in $P_i[b'_i,d_i]$ and v is in $P_{i+1}[a_{i+1},c'_{i+1}]$. Since $(\{u,v\},\alpha)$ is not a pin, there exists a link $\widehat{pq}$ with p coming before u on P_i and q coming after v on P_{i+1}. But then the directed cycle $\widehat{pq}P_{i+1}]q,d_{i+1}]P_{i+3}...P_i[a_i,p]$ is disjoint from the undirected cycle $P_i[u,d_i]P_{i+2}...P_{i+1}[a_{i+1},v]\alpha u$ (see right part of Figure 5), certifying that $(\{u,v\},\alpha)$ is a 1-clasp.

Now consider the case that one of u,v, say, u, is not on a wall and, hence, an internal vertex of a link $\widehat{u_1u_2}$ between two consecutive walls. Define similarly v_1,v_2 if v is not on a wall, and $v_1=v_2=v$ otherwise. There is always a couple (u_g,v_h), with $g,h \in \{1,2\}$, such that u_g and v_h are on the same or on distinct non-consecutive walls. If u_g and v_h are on the same wall P_i then the subdigraph induced by $UG(D[V(P_i) \cup \{\alpha\} \cup V(\widehat{u_1u_2})])$ contains a cycle which avoids all walls except for P_i, and hence, by Lemma 3, $(\{u,v\},\alpha)$ is a 1-clasp.

If u_g and v_h are on distinct non-consecutive walls then we can relabel everything in the same way as in the proof of Lemma 4 having one of u_g,v_h on P_0 and the other on P_{2k}, where $0<2k<\ell-1$. If u_g in P_0 and v_h in P_{2k} then $u,...,u_g,R,v_h,...,v,\alpha,u$ — where R is the path joining u_g and v_h through walls $P_0,P_2,\ldots,P_{2k}$ — forms a cycle in $UG(D)$ disjoint from any directed cycle contained in the subdigraph induced by $P_1,P_3,...,P_{\ell-2},P_{\ell-1}$ and all the links between them. Otherwise, $v,...,v_h,R,u_g,...,u,\alpha,v$ — where R is the path joining v_h and u_g through walls $P_0,P_2,\ldots,P_{2k}$ — is the desired cycle. $\square$

Theorem 7 *There is a polynomial time algorithm that decides whether a given digraph D whose unique nontrivial strong component is a subdivision of a vault has a dicycle B in D and a cycle C in $UG(D)$ with $V(B) \cap V(C) = \emptyset$, and finds these cycles if they exist.*

Proof. We first reduce to the situation described immediately before Lemma 3. For every $\alpha \in D - D'$ consider the sets $\{u, v\}$ formed by two distinct neighbors u, v of α. For each such $(\{u, v\}, \alpha)$ it takes polynomial time to check if $(\{u, v\}, \alpha)$ is a pin (according to (i) of Theorem 6 this is equivalent to check whether $D' - \{u, v\}$ is acyclic). As soon as $(\{u, v\}, \alpha)$ is not a pin, one gets the two cycles as in the proof of Theorem 6. If all $(\{u, v\}, \alpha)$ turn out to be pins, then there is no 1-clasp by (ii) of Theorem 6, and hence D is a no-instance. $\square$

Multiwheels and split multiwheels.

Assume now that D' is a subdivision of a multiwheel or split multiwheel, with central vertices a or a^-, a^+, respectively. If D' is a multiwheel then set $A' := \{a\}$, otherwise define A' to be the set of vertices of the link $\widehat{a^- a^+}$. Let B' be the set of internal vertices of the links with exactly one end vertex in A'. Let $C' := D' - (B' \cup A')$ be the remaining cycle. It is quite simple to list all the dicycles of D', so that brute force works.

Theorem 8 *There is a polynomial time algorithm that decides whether a given digraph D whose unique nontrivial strong component is a subdivision of a multiwheel or of a split multiwheel has a dicycle B in D and a cycle C in $UG(D)$ with $V(B) \cap V(C) = \emptyset$, and finds these cycles if they exist.*

Proof. A dicycle in D' is either C', or it is formed by two links $\widehat{ca}, \widehat{a'c'}$ with $a, a' \in A$ and $c, c' \in V(C')$ together with the unique (a, a')-path in $D'[A']$ and the unique (c', c)-path in C'. Hence there are only $O(|V(D')|^2)$ many dicycles, and for each such dicycle B we check if $UG(D) - V(B)$ contains a cycle. This leads straightforwardly to a cubic time algorithm as desired. $\square$

Trivaults.

Assume now that D' is a subdivision of a trivault D'_0, and let L_i, R_i, b_i, c_i for $i \in \{0, 1, 2\}$ be as in the definition of a trivault (with D'_0 instead of D). Again, we have good control on the dicycles:

Theorem 9 *There is a polynomial time algorithm that decides whether a given digraph D whose unique nontrivial strong component is a subdivision of a trivault has a dicycle B in D and a cycle C in $UG(D)$ with $V(B) \cap V(C) = \emptyset$, and finds these cycles if they exist.*

Proof. Set $X_i := L_i \cup R_i$ for $i \in \{0, 1, 2\}$. If a dicycle in D'_0 contains a vertex of X_i then it enters X_i via an arc from some vertex from R_j with $j \neq i$ to

some $\ell \in L_i$, and it exits X_i via an arc from some $r \in R_i$ to some vertex from L_k with $k \neq i$. Moreover, the dicycle will contain the unique ℓ, r-path in X_i and, in particular, b_i and c_i — hence it cannot traverse X_i more than once. Therefore, every dicycle in D is formed by either (i) a pair $(a, b), (c, d)$ of arcs with $a \in R_i, b \in L_j, c \in R_j, d \in L_i$, where $i \neq j$ together with the unique (b, c)-path in X_j and the uniqe (d, a)-path in X_i, or (ii) a triple $(a, b), (c, d), (e, f)$ with $a \in R_0, b \in L_1, c \in R_1, d \in L_2, e \in R_2, f \in L_0$ together with the unique (b, c)-path in X_1, the unique (d, e)-path in X_2, and the unique (f, a)-path in X_0, or (iii) a triple $(a, b), (c, d), (e, f)$ with $a \in R_0, b \in L_2, c \in R_2, d \in L_1, e \in R_1, f \in L_0$ together with the unique (b, c)-path in X_2, the unique (d, e)-path in X_1, and the unique (f, a)-path in X_0. As the dicycles in D' are obtained by those in D'_0 by replacing arcs with the respective links, there are only $O(|E(D'_0)|^3)$ many dicycles in D, and we can construct them easily. For each such dicycle B we check if $UG(D) - V(B^*)$ contains a cycle. This leads straightforwardly to a $O(|V(D)|^8)$-time algorithm as desired. $\square$

4 Digraphs D with $\tau(D) = 1$

The aim of this section is to prove that Problem 1 is $\mathcal{NP}$-complete for digraphs with transversal number 1 and an unbounded number of transversal vertices. We start with a quite different $\mathcal{NP}$-complete problem on bipartite graphs and then show how to reduce from this problem.

Problem 2 *Let G be a 2-connected bipartite graph with color classes U and V and let $V_1, V_2, \ldots, V_k$ be a partition of V into disjoint non-empty sets. Decide if there exists a cycle C in G which avoids at least one vertex from each V_i.*

Lemma 5 *Problem 2 is $\mathcal{NP}$-complete.*

Proof. We will show how to reduce 3SAT to Problem 2 in polynomial time. Let $W[u, v, p, q]$ be the graph with vertices $\{u, v, y_1, y_2, \ldots, y_p, z_1, z_2, \ldots, z_q\}$ and the edges of the two (u, v)-paths $uy_1y_2 \ldots y_pv$ and $uz_1z_2 \ldots z_qv$. Graphs of this type will form the *variable gadgets.*

Let $\mathcal{F}$ be an instance of 3SAT with variables $x_1, x_2, \ldots, x_n$ and clauses C_1, C_2, …, C_m. We may assume without loss of generality that each variable x occurs at least once in either the negated or the non-negated form in $\mathcal{F}$. The ordering of the clauses $C_1, C_2, \ldots, C_m$ induces an ordering of the occurrences of a variable x and its negation $\bar{x}$ in these. With each variable x_i we associate a copy of $W[u_i, v_i, 2p_i + 1, 2q_i + 1]$ where x_i occurs p_i times and $\bar{x}_i$ occurs q_i times in the clauses of $\mathcal{F}$. Initially, these copies are assumed to be disjoint, but we chain them up by identifying v_i and u_{i+1} for each $i \in \{1, 2, \ldots, n-1\}$. Let $s = u_1$ and $t = v_n$. Let G' be the graph obtained in this way. Observe that G'

is bipartite since each $W[u_i, v_i, 2p_i + 1, 2q_i + 1]$ is the union of two even length (u_i, v_i)-paths.

For each $i \in \{1, 2, \ldots, m\}$ we associate the clause C_i with three of the vertices $V_i = \{a_{i,1}, a_{i,2}, a_{i,3}\}$ (this is the *clause gadget*) from the graph G' above as follows: assume C_i contains variables x_j, x_k, x_ℓ (negated or not). If x_j is not negated in C_i and this is the rth occurence of x_j (in the order of the clauses that use x_j), then we identify $a_{i,1}$ with $y_{j,2r-1}$ and if C_i contains $\bar{x}_j$ and this is the hth occurrence of $\bar{x}_j$, then we identify $a_{i,1}$ with $z_{j,2h-1}$. We proceed similarly with $x_j, a_{i,2}$ and $x_k, a_{i,3}$, respectively. Thus G' contains all the vertices $a_{j,i}$, $j \in \{1, \ldots, m\}, i \in \{1, 2, 3\}$.

Claim. G' contains an (s, t)-path P which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in \{1, \ldots, m\}$ if and only if $\mathcal{F}$ is satisfiable.

For a proof, suppose P is an (s, t)-path which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in \{1, \ldots, m\}$. By construction of G', for each variable x_i, P traverses either the subpath $u_i y_{i,1} y_{i,2} \ldots y_{i,2p_i+1} v_i$ or the subpath $u_i z_{i,1} z_{i,2} \ldots z_{i,2q_i+1} v_i$. Now define a truth assignment by setting x_i false if and only if the first traversal occurs for i. This is a satisfying truth assignment for $\mathcal{F}$ since for any clause C_j at least one literal is avoided by P and hence becomes true by the assignment (the literals traversed become false and those not traversed become true). Conversely, given a truth assignment for $\mathcal{F}$ we can form P by routing it through all the false literals in the chain of variable gadgets. This proves the claim.

Now let B be the bipartite graph with color classes U, V which we obtain from G' by adding new vertices z_1, z_2 and the edges sz_1, sz_2, z_1t, z_2t. Here V is the vertex set $\{z_1, z_2\} \cup \{y_{i,2j+1} : i \in \{1, \ldots, m\}, j \in \{1, \ldots, p_i\}\} \cup \{z_{i,2j+1} : i \in \{1, \ldots, m\}, j \in \{1, \ldots, q_i\}\}$, and U is the set of the remaining vertices. For each $i \in \{1, \ldots, m\}$ let $V'_i = \{y_{i,2p_i+1}, z_{i,2q_i+1}\}$ and let $V_{m+1} = \{z_1, z_2\}$. Then $V_1, V_2, \ldots, V_m, V'_1, \ldots, V'_m, V_{m+1}$ form a partition of V.

It is clear from the construction of G that every cycle C distinct from the 4-cycle sz_1tz_2s is either formed by one of the subgraphs $W[u_i, v_i, 2p_i + 1, 2q_i + 1]$ or consists of an (s, t)-path in G and one of the two (t, s)-paths tz_1s, tz_2s.

We show that G has a cycle C which avoids at least one vertex from each of the sets $V_1, V_2, \ldots, V_m, V'_1, \ldots, V'_m, V_{m+1}$ if and only if $\mathcal{F}$ is satisfiable. This follows from our claim and the fact that the definition of V'_i, $i \in \{1, \ldots, m\}$, and V_{m+1} implies that the desired cycle exists if and only if G' has an (s, t)-path which avoids at least one vertex from $V_j = \{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in \{1, \ldots, m\}$. Note that the sets V'_i, $i \in \{1, \ldots, m\}$, exclude cycles of the form $W[u_i, v_i, p_i, q_i]$ and V_{m+1} excludes the cycle sz_1tz_2s. □

We now reduce Problem 2 to Problem 1 restricted to the case of dicycle transversal number 1 and an unbounded number of transversal vertices.

Let H be a bipartite graph with color classes U, V where $U = \{b_1, ..., b_r\}$, and $V = V_1 \cup V_2 \cup \ldots \cup V_k$ with $V_i = \{p_{i,1}, ..., p_{i,\ell_i}\}$, $\ell_i > 0$, and $V_i \cap V_j = \emptyset$ if $i \neq j$.

We form a directed graph D in the following way: Create $k+1$ vertices v_0, $v_1, \ldots, v_k$ (each but the first representing some V_j). Create vertices $p_{i,j}, b_\ell$ for each $p_{i,j}, b_\ell$ of the bipartite graph. Create the arcs $v_{i-1}p_{i,j}$ and $p_{i,j}v_i$ for all $i \in \{1, ..., k\}$, $j \in \{1, ..., \ell_i\}$. Create an arc $b_\ell p_{i,j}$ for each edge $b_\ell, p_{i,j}$ of the bipartite graph. Finally, add the arc $v_k v_0$.

Lemma 6 *D contains a dicycle B and a cycle C of $UG(D)$ which are disjoint if and only if there is a cycle in H avoiding a vertex of V_i for each i.*

Proof. First suppose there is a cycle in H avoiding the vertex p_{i,a_i} of V_i for each i. Then, by the construction of D, the same cycle will be a cycle in $UG(D)$. The cycle $v_0 p_{1,a_1} v_1 p_{2,a_2} ... v_{k-1} p_{k,a_k} v_k v_0$ is vertex disjoint from this undirected cycle, and we are done.

Now suppose there is an undirected cycle C disjoint from some dicycle in D. Note that every dicycle in D is formed by the arc $v_k v_0$ and some (v_0, v_k)-path. The path is of the form $v_0 p_{1,a_1} v_1 ... v_{k-1} p_{k,a_k} v_k$. Hence C does not contain any of the vertices $v_0, v_1, \ldots, v_k$ and hence uses only $p_{i,j}$ or b_ℓ vertices and always alternates between them. Therefore C has a corresponding cycle in H, and this one avoids at least the vertex p_{i,a_i} from of the set V_i for each $i \in \{1, \ldots, k\}$. $\square$

From the previous two lemmas we immediately get:

Theorem 10 *Problem 1 is $\mathcal{NP}$-complete.*

5 Digraphs D with $\tau(D) = 1$ and a bounded number of dicycle transversals

Consider a digraph D with $\tau(D) = 1$. We show that if there is a bounded number of transversal vertices then our problem is polynomially decidable. We start by deleting each arc connecting a transversal vertex with an external vertex. These will never be used to certify a yes-instance because every transversal vertex is contained in the directed cycle. After this process we delete external vertices with degree at most 1.

Let C be a dicycle of D and let $a, a_1, \ldots, a_{k-1}$ be the transversal vertices of D, in the order they show up on the cycle. Build a new acyclic digraph $\tilde{D}$ by splitting a into an outgoing part a_0 and an ingoing part a_k. All arcs leaving (entering) a now leave a_0 (enter a_k). Given the preprocessed graph our problem is equivalent to that of finding in $\tilde{D}$ a directed (a_0, a_k)-path disjoint from an undirected cycle. Note that all transversal vertices are (a_0, a_k)-separators in $\tilde{D}$, and every (a_0, a_k)-path contains $a_0, a_1, \ldots, a_k$ in that order. For $x \in \{1, \ldots, k\}$,

fix a largest system $\mathcal{P}^x$ of openly disjoint (a_{x-1}, a_x)-paths, say, $P^x_1, \ldots, P^x_{\ell^x}$, and let $P^* := \bigcup_{x=1}^{k} \bigcup_{i=1}^{\ell_x} P^x_i$ be the digraph formed by the union of all these paths. Note that no vertex except $a_1, \ldots, a_{k-1}$ belongs to more than one system $\mathcal{P}^x$.

Now suppose that there exists an (a_0, a_k)-dipath C in $\tilde{D}$ and a cycle C' in $UG(\tilde{D})$ disjoint from C. We show that we can take them such that C changes from one path to another at most once in any of the path systems. In fact, we can take C, C' as above such that the number of their arcs not in the path system, that is,

$$|A(C \cup C') \setminus A(P^*)|, \tag{1}$$

is minimized. For all paths P^x_i as defined above, let $Q^x_{i,1}, ..., Q^x_{i,h^x_i}$ be the connected components of $C \cap P^x_i$ ordered such that $Q^x_{i,j}$ is before $Q^x_{i,j'}$ on P^x_i if $j < j'$. Likewise, let $R^x_{i,1}, ..., R^x_{i,k^x_i}$ be the connected components of $P^x_i \setminus C$, if any, ordered in the same way as before. Let $b^x_{i,j}$ and $c^x_{i,j}$ be the first and the last vertex of $Q^x_{i,j}$, respectively. With this notation we have $a_x = b^x_{i,1}$ and $a_{x+1} = c^x_{i,h^x_i}$ for all i.

Claim 1. For all x, i, the dipath C visits $Q^x_{i,1}, ..., Q^x_{i,h^x_i}$ in this order.

For if C would first visit $Q^x_{i,j'}$ and then $Q^x_{i,j}$, with $j < j'$, then $\tilde{D}$ contained the dicycle $C[b^x_{i,j'}, b^x_{i,j}]P^x_i[b^x_{i,j}, b^x_{i,j'}]$, contradiction. This proves Claim 1.

Claim 2. For all x, i, j, $d_{C'}(R^x_{i,j}) = 2$. [4]

For a proof, observe that $d_{C'}(R^x_{i,j})$ is even, and positive, for otherwise, by replacing the $(b^x_{i,j}, b^x_{i,j+1})$-subpath of C by $P^x_i[b^x_{i,j}, b^x_{i,j+1}]$ we get an (a_0, a_k)-path which is still disjoint from C' but gives a lower value for (1). Now let s and t be the first and last vertex on $R^x_{i,j}$ from C', respectively. If $d_{C'}(R^x_{i,j}) \geq 4$, then the digraph induced by $V(C') \cup V(R^x_{i,j})$ contains a cycle C'' such that replacing C' by C'' yields a lower value for (1). This proves Claim 2.

Claim 3. P^x_i does not contain the arc $c^x_{i,j} b^x_{i,j+1}$.

For if it would then we could replace the $(c^x_{i,j}, b^x_{i,j+1})$-subpath of C by this arc and get, again, a smaller value for (1). This proves Claim 3.

We define a *bridge* as the subdigraph of $\tilde{D}$ formed by either a single arc of $A(\tilde{D}) - A(P^*)$ connecting two vertices of P^*, or the arcs incident with the vertices of a connected component of $UG(\tilde{D} - V(P^*))$. We may assume that a bridge neither contains two interior vertices of any P^x_i nor a cycle of $UG(\tilde{D})$, for if it would then we easily find a dipath C and C' with a smaller value for (1).

A *switch* is a maximal subpath of C of length at least one such that all its edges and internal vertices belong to some bridge. It is then evident that a switch is a (v, w)-subpath of a single bridge where v is contained in some P^x_i and w is

[4]For a subdigraph H of a digraph D, let $d_D(H)$ denote the number of edges in D having exactly one end vertex in H.

contained in some P_j^y. Since $\tilde{D}$ is acyclic, $y \geq x$, but if $y > x$ then C misses a_x, contradiction. Hence $x = y$, and we call the switch, more specifically, an *x-switch.* We may achieve that $i \neq j$, for suppose that v, w are both from P_i^x. If P_i^x was the only path in $\mathcal{P}^x$ then it has length one (for otherwise, some internal vertex would separate a_{x-1} from a_x by MENGER's Theorem and the maximality of $|\mathcal{P}^x|$); but then $v = a_{x-1}$ and $w = a_x$, so that our switch is openly disjoint from P_i^x, contradicting again the maximality of $|\mathcal{P}^x|$. So $\mathcal{P}^x$ contains at least two paths P_i^x, P_j^x, $i \neq j$ — and since not both v, w are internal vertices of P_i^x, we may assume that at least one of v, w is on P_j^x, too.

Claim 4. For every x, there is at most one x-switch.

For suppose, to the contrary, there are at least two, and consider the first two along C. Suppose the first one is from P_i^x to P_j^x, where $i \neq j$. Then the second one is from P_j^x to some P_k^x. By Claim 3, $P_i^x \setminus C$ has at least one nonempty component, so consider $R_{i,1}^x$, and $P_j^x \setminus C$ has at least two, so consider $R_{j,1}^x$ and $R_{j,2}^x$. By Claim 2, exactly one of the two $(R_{j,1}^x, R_{j,2}^x)$-subpaths of C' misses $R_{i,1}^x$ (for otherwise $d_{C'}(R_{i,1}^x) \neq 2$). Let us denote this by path by M. But then one could change C using $P_i^x[b_{i,1}^x, b_{i,2}^x]$ instead of $C[b_{i,1}^x, b_{i,2}^x]$, which contains $Q_{j,2}^x$. Now $M \cup R_{j,1}^x \cup R_{j,2}^x \cup Q_{j,2}^x$ contains an undirected cycle C' disjoint from the new dicycle C, and together they achieve a lower value for (1). This contradiction proves Claim 4.

Theorem 11 *For fixed k, there is a polynomial time algorithm that decides whether a given digraph D with $\tau(D) = 1$ and at most k dicycle transversal vertices has a dicycle B in D and a cycle C in $UG(D)$ with $V(B) \cap V(C) = \emptyset$, and finds these cycles if they exist.*

Proof. The problem is (polynomially) equivalent to finding C, C' in $\tilde{D}$ as in the first three paragraphs of this section (or decide that they do not exist). All further objects, in particular suitable maximal path systems $\mathcal{P}^x$, can be computed in polynomial time, and the considerations including Claim 4 guarantee that there are C, C' as desired if and only if there are C, C' as desired with at most one x-switch for each x.

We first iterate through all k-tuples $\pi = (\pi_1, \ldots, \pi_k)$, where, for each x, π_x is a path from $\mathcal{P}^x$. There are less than $|V(D)|^k$ choices. For each π, set $C_\pi := \bigcup_{x=1}^k \pi_k$ and check if $UG(\tilde{D} \setminus C_\pi)$ has a cycle C'. All that can be done in polynomial time, and we stop (with a yes-instance) as soon as we find such a C'.

Now we are in a stage where a solution would use at least one switch. However, at the same time, we have control on the number of hypothetical x-switches and can determine these. For all pairs (e, f) of arcs we check if e starts on some P_i^x and f ends in some P_j^x and if there is a dipath starting with e and ending with f without internal vertices from $V(P^*)$. This can be done in polynomial time, and such a path is uniquely determined because otherwise there would

be a cycle C' in $UG(\tilde{D} \setminus P^*)$, which we would have detected while iterating through the π earlier as above. Such a path might serve as an x-switch for more than one pair of paths P_i^x, P_j^x if one and hence only one of its end vertices is a transversal vertex; we can maintain a list of the options for each of them and this list has lenght at most $|V(D)|$. The number of hypothetical x-switches for each x is thus bounded by $|A(D)|^2$, hence we find all of them, plus their lists, in polynomial time.

Now we iterate through all k-tuples $\pi = (\pi_1, \dots, \pi_k)$, where, for each x, π_x is either a path from $\mathcal{P}^x$ or a hypothetical x-switch connecting P_i^x, P_j^x with $i \neq j$. (Moreover, we may assume that not all of the π_x are paths from $\mathcal{P}^x$, as such a π has been considered earlier above.) There are far less than $(|A(D)|^2 + 1)^k$ choices for π here. For each π, construct a dipath C_π as follows: For each hypothetical x-switch π_x, say, starting at u and ending at v, take its union with the unique (a_{x-1}, u)- and the unique (v, a_x)-path in $\bigcup_{i=1}^{\ell_x} P_i^x$. Take the union of all these paths and of those π_x which have been selected as paths from $\mathcal{P}^x$ and call it C_π. It is clear that if C, C' as desired exist then $C = C_\pi$ for some π. Hence it suffices to check if $\tilde{D} \setminus C_\pi$ has a cycle C', for all C_π. All that can be done in polynomial time. □

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